

**Ohio Section of the MAA
Student Team Competition
March 26, 2004**

NUMBER 1

Let $a_1, a_2 \in \mathbb{R}$ and define $a_n = a_{n-1} + a_{n-2}$ for $n = 3, 4, \dots, 10$.
Find the sum

$$\sum_{i=1}^{10} a_i$$

as a function of a_7 . Prove your result.

NUMBER 2

A sequence (F_n) is defined by the recurrence relation

$$F_{n+1} = 3F_n + 4F_{n-1}, \quad F_0 = 1, \quad F_1 = 1$$

Find the units digit of F_{999} .

NUMBER 3

Find the determinant of the $n \times n$ matrix

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

NUMBER 4

Let $f(x)$ be a continuous function with the property that

$$f(x + y) = f(x)f(y)$$

for all real numbers x and y . If $f(1) = 9$, prove that $f(x) = 9^x$ for all x .

NUMBER 5

Let A be the matrix below. Find the characteristic polynomial of A and the eigenvalues of A . For each eigenvalue, find an eigenvector. Then find a matrix, P , that will diagonalize A .

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

NUMBER 6

Evaluate

$$\int \frac{1}{(x-2)^{1/2} + (x-2)^{2/3}} dx$$

NUMBER 7

Let $a, n > 1$ be two integers. Show that the number of integers in the list

$$1, \dots, a^n - 1$$

that are relatively prime to $a^n - 1$ is a multiple of n .

NUMBER 8

Find all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ with the property

$$\int_{x^2}^x f(t) dt = x - x^2$$

for all $0 \leq x \leq 1$.

NUMBER 9

Find $a > 0$ so that the following function is a probability density function for a continuous random variable X :

$$f(x) = \begin{cases} \sqrt{a^2 - (x - a)^2} & 0 \leq x < a \\ \frac{1}{2}|x - 3a| & a \leq x \leq 5a \\ \sqrt{a^2 - (x - 5a)^2} & 5a < x \leq 6a \end{cases}$$

NUMBER 10

Let $f_1(x) = f(x) = \frac{1+x}{1-x}$, and for $n > 1$, $f_n(x) = f(f_{n-1}(x))$. Find $f_{2003}(2004)$

TIEBREAKER

It can be shown that for every real number y there exists an integer m and integers $d_k \in \{-1, 0, 1, 7, 23\}$ such that

$$y = \sum_{k=-\infty}^m d_k 5^k$$

Notice that this is just a base 5 expansion of y using the digits $\{-1, 0, 1, 7, 23\}$. For example, $\frac{158}{25}$ can be written in this base as

$$\frac{158}{25} = 1(5)^1 + (-1)(5)^0 + 7(5)^{-1} + 23(5)^{-2} = (1, -1.7, 23)_5$$

The number $\frac{23}{4}$ has at least five different representations in this base. Find as many as you can.